SPRING 2024: BONUS PROBLEM 1 SOLUTION

For the following bonus problem, you will receive up to 10 points added to your course total. Since this is a bonus problem, your answer must be essentially completely correct to receive bonus points. Please upload your solution **no later than** 6pm on Thursday, February 22.

Suppose V is a vector space over $\mathbb{R}$ with basis $\alpha := \{v_1, \ldots, v_n\}$. Let $\{u_1, \ldots, u_n\}$ be vectors in V. Prove that the following statements are equivalent.

(i) $\{u_1, \ldots, u_n\}$ is a basis for V.
(ii) $\{[u_1]_\alpha, \ldots, [u_n]_\alpha\}$ is a basis for $\mathbb{R}^n$.
(iii) The matrix A whose columns are $[u_1]_\alpha, \ldots, [u_n]_\alpha$ is invertible.

Solution. For (i) implies (ii): Let $v_0 := \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ be an arbitrary vector in $\mathbb{R}^n$. Set $v := a_1v_n + \cdots + a_nv_n \in V$. Then $v = b_1u_1 + \cdots + b_nu_n$, for $b_i \in \mathbb{R}$, by our assumption (i). Thus,

$$v_0 = [v]_\alpha = [b_1u_1 + \cdots + b_nu_n]_\alpha = b_1[u_1]_\alpha + \cdots + b_n[u_n]_\alpha,$$

which shows that the vectors $\{[u_1]_\alpha, \ldots, [u_n]_\alpha\}$ span $\mathbb{R}^n$. Suppose $c_1[u_1]_\alpha + \cdots c_n[u_n]_\alpha = \vec{0}$ in $\mathbb{R}^n$. Then $[c_1u_1 + \cdots + c_nu_n]_\alpha = \vec{0}$, which implies $c_1 + \cdots + c_nu_n = \vec{0}$ in V. By the assumption (i), $c_1 = \cdots = c_n = 0$, showing that $\{[u_1]_\alpha, \ldots, [u_n]_\alpha\}$ are linearly independent, and thus form a basis for $\mathbb{R}^n$.

Note: Since $\{[u_1]_\alpha, \ldots, [u_n]_\alpha\}$ are n vectors in $\mathbb{R}^n$ it suffices to prove just one of the condition required of a basis. I included proofs of both to show how to prove each one.

For (ii) implies (iii): Let $e_i \in \mathbb{R}^n$ denote the ith standard basis vector. Then we can write

$$e_i = b_{i1}[u_1]_\alpha + \cdots + b_{in}[u_n]_\alpha.$$

This means $A \cdot \begin{pmatrix} b_{i1} \\ \vdots \\ b_{in} \end{pmatrix} = e_i$. Thus, if we set $B := (b_{ij})$, we have $AB = I_n$, the $n \times n$ identity matrix. Strictly speaking, one should also show $BA = I_n$, but if you got this far, that is good enough.

For (iii) implies (ii): We first note that the columns of A form a basis for $\mathbb{R}^n$. To see this, let $C_1, \ldots, C_n$ be the columns of A. Suppose $v \in \mathbb{R}^n$. Set $w = A^{-1}v$ and suppose $w = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$. Then

$$c_1C_1 + \cdots + c_nC_n = Aw = A(A^{-1}v) = v.$$

This shows the columns of A span $\mathbb{R}^n$. Since these are n vectors in $\mathbb{R}^n$, they form a basis for $\mathbb{R}^n$. Now suppose $a_1u_1 + \cdots + a_nu_n = \vec{0}$ in V. Then in $\mathbb{R}^n$,

$$\vec{0} = [a_1u_1 + \cdots + a_nu_n]_\alpha = a_1[u_1]_\alpha + \cdots + a_n[u_n]_\alpha.$$

Since $\{[u_1]_\alpha, \ldots, [u_n]_\alpha\}$ are linearly independent, $a_1 = \cdots = a_n = 0$, showing that $u_1, \ldots, u_n$ are linearly independent. Thus, these vectors form a basis for V.